INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2003; **43**:1167–1176 (DOI: 10.1002/fld.545)

Meshless numerical simulation of (full) potential flows in a nozzle by genetic algorithms

G. Winter*,†, J. C. Abderramán, J. A. Jiménez, B. González, E. Benitez and P. Cuesta

Institute of Intelligent Systems and Numerical Applications in Engineering (IUSIANI), Universidad de Las Palmas de Gran Canaria, Campus Universitario, 35017 Las Palmas, Spain

SUMMARY

A new procedure to solve some fluid problems formulated in elliptical partial differential equations is presented. A Genetic Algorithm with a dynamical encoding and a partial grid sampling is proposed for it as the advantages of solving the problem without using all grid nodes at the same time, and of adjusting step grid, without increasing the complexity. The designed method has immediate applications some self-contained and some in combination with other traditional methods. Also, it provides a method alternative to the existing ones and uses simpler operations. Theoretical mathematical foundations of the problem are easily incorporated and that as a powerful characteristic of the method. In practice, our focus is to obtain an acceptable approximated solution. The method makes it possible to solve problems with vague boundary conditions since no algebraic equation system is involved in the process. From the solution reached we have good information available to make an appropriate mesh to solve the problem through a traditional method. Comparative results for both linear and non-linear potential flow problems inside a nozzle are given. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: compressible flow; evolutionary algorithms; genetic algorithms; meshless heuristic simulation; nozzle flows

1. INTRODUCTION

Numerical solutions approximated by traditional methods such as finite differences (FD), finite elements and finite volumes are often used to solve Engineering and Applied Physics problems formulated in partial differential equations (PDE). The solution quality obtained from most FD traditional schemes depends remarkably on the characteristic of the grid, on the number of nodes. To obtain good results, it is necessary to use grids with a greater number of nodes with a small distance among them. In this way the computational cost is increased considerably due to the size of the linear equation systems to be solved.

* Correspondence to: G. Winter, IUSIANI, Universidad de Las Palmas de Gran Canaria, Campus Universitario, 35017 Las Palmas, Spain.
† E-mail: gabw@step.es, web page: http://ceani.ulpgc.es/

Received 3 March 2002
Accepted 19 February 2003

The non-linear problems show even more difficulties, and a stronger effort is needed to reach solution at all points of the grid simultaneously. Any alternative approach must exhibit some or all of the following advantages:

- Possibility to solve the problem with a more local point of view, for example, by using domain decomposition or partial samples of subsets of nodes of the total grid, i.e. not using the whole grid at the same time.
- Use of simpler operations, which results in a smaller complexity.
- Capacity to vary step grid without increasing the complexity meaningfully.
- Easy adaptability to parallelize the algorithms.

The Evolutionary Algorithms (EAs) [1] are search procedures which have been used in many real world applications because they work somewhat independent of the problem characteristics. Among them, the Genetic Algorithms (GAs) [2, 3], can be used in combination with FD schemas to optimise them to increase the efficiency of this method [4] and it can be a tool in computational fluid dynamical field also. In the present paper, we propose a methodology that uses GAs to obtain fluid speeds for both potential and full potential flows, in a curvilinear duct and in a nozzle. To solve this kind of problem with evolutionary strategies it is necessary to start from an initial value of the solution domain for each point that permits GA to evolve towards a solution nearby the desirable one. Furthermore, an adequate fitness function is necessary.

The information based on the knowledge of the qualitative behaviour of the solution and some simple qualitative properties, permits us to establish an adequate initial solution. Through a dynamical encoding for the candidate solutions, we can move towards a solution with the same or very nearly the same quality as the one obtained by other methods. In linear or quasi-linear cases, the qualitative properties of the solution are obtained from the mathematical analysis of the PDE and these can be considered in the search process in a simple way.

2. GENETIC ALGORITHM WITH DYNAMICAL ENCODING AND PARTIAL GRID SAMPLING

The general characteristics of our evolutionary algorithm proposed here are discussed below. A grid is built and the GA tests a population of candidate solutions through a partial grid sampling [4] of the nodes of the grid in each generation. These $U(x, y)$ solutions for each node are sampled from a dynamic interval. The aim of verifying a numerical scheme at each nodal point is equivalent to minimizing the corresponding objective function. By using the tournament between the candidate and the previous best solution, the selection in one node is carried out. One point crossover and a smooth mutation are used in the exploration stage. The pseudocode for the whole evolutionary algorithm is illustrated in Figure 1.

2.1. Partial grid sampling

In each test case, a grid is built in the domain. An advantage of the present method is the possibility of using only a sample of nodes solution at each generation instead of the whole grid. The partial grid sampling is based on choosing j different nodes randomly at each generation. Each chromosome in the population corresponds to one different node solution in


```
The grid n X m is built in the domain
Initial values for U(x,y) are built: U1(x,y), ..., Unxm(x,y)
For q=1, generations number
  j nodes are chosen randomly from the grid: {node j}
   the j nodes are mapping on the chromosomes by encoding
      Uj(x,y) in  node j <=> chromosome j
These j chromosomes form the population
   1-point crossover operator acts on the population
   smooth mutation operator acts on the population
   each chromosome is decoded as a candidate U*(x,y)
      chromosome j => U*j(x,y) in node j
   selection operator f acts on U*(x,y)
   for p=1, j
   if  f [U*p(x,y)] is less than f [Up(x,y)] then
          Up(x,y) = U*p(x,y)
   end for
    end for
```

Figure 1. The pseudocode for the whole evolutionary algorithm.

the grid. The chromosomes give the value of the solution in its node of the grid. All nodes must have been chosen before a node can be chosen again. In successive generations, the nodes are chosen to complete the whole grid.

2.2. Dynamical encoding

A dynamical encoding [5] is used to calculate the $U(x, y)$ from the chromosomes. In that way, for the node (x_j, y_j) and the chromosome integer value $i \in [0, 2^l - 1]$, the $U(x_j, y_j)$ is

$$\begin{aligned} U(x_j, y_j) &= i \cdot \frac{b-a}{2^l - 1} + a \\ b &= \max\{U(x_j, y_{j+1}), U(x_j, y_{j-1})\} \\ a &= \min\{U(x_j, y_{j+1}), U(x_j, y_{j-1})\} \end{aligned} \tag{1}$$

where $U(x_j, y_{j+1})$ is the value of $U(x, y)$ at the upper node (x_j, y_{j+1}), $U(x_j, y_{j-1})$ is the value of $U(x, y)$ at the lower node (x_j, y_{j-1}) and l is the chromosome length. The previous method is valid for either increasing or decreasing the subdomain in the space of solutions.

This encoding is dynamical because the $U(x, y)$ changes as time passes in the evolutionary process. So, the fitness distance between nodes changes too. This dynamic encoding permits the GA to work in good search regions and save computational cost.

2.3. Genetic operators

1-*point crossover*: $\chi = 1$: It is the Holland's classical genetic operator [2].

Smooth mutation: A mutation smooth enough is used [4] in both genotype and phenotype space. The smooth mutation process is:

1. A random number, rand, is chosen.
2. If rand $\leqslant \mu$ (mutation probability) the individual mutates, otherwise it does not.
3. If rand $\in (\mu^{k-1}, \mu^k]$ then k mutations are accomplished, $k = 1, 2, 3, \ldots, L$. (A bit can mutate more than once). Here μ^k is the kth power of μ.
4. If standard binary codification is used, the most left k bits in each criterion (co-ordinate) do not mutate. The $l - k$ remaining bits can mutate from 0 to k times.
5. While the whole population is not checked go to 1.

This type of mutation is a little disruptive, and for this reason it is necessary to use high values of mutation probabilities (0.7–0.8). With the fourth step the smooth effect of the mutation is increased.

Selection: A tournament selection operator between the new candidate and the previous solution is accomplished. The fitness function is a numerical scheme.

3. EVOLUTIONARY FLUIDS DYNAMICS

3.1. Potential flow in a duct

We introduced this evolutionary method in Fluid Dynamics with the resolution of the potential flow in a curvilinear duct [6, 7]. The streamfunction $U(x, y)$ accomplishes the following linear PDE:

$$\begin{gathered}
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0; \quad \text{with } \Omega = \{x, y\colon\ 0 \leqslant x \leqslant 2;\ 0 \leqslant y \leqslant H(x)\} \subset \Re^2 \\
H(x) = \frac{3}{8} + \frac{1}{8}\sin\left[\pi\left(x + \frac{1}{2}\right)\right] \\
U(x, 0) = 0; \quad U(x, H(x)) = 1 \quad \text{and} \quad \left(\frac{\partial U}{\partial x}\right)_{(0, y)} = \left(\frac{\partial U}{\partial x}\right)_{(2, y)} = 0
\end{gathered} \tag{2}$$

The streamfunction iso-values in the duct coincide with the curves in $\Re^2$. This function accomplishes the equation $U(x, y) = \eta = \text{constant}$, and the solution agrees with the solution found for the EDP. The grid has been built taking advantage of this information (Figure 2). The points (x, y) that fulfil the equation are considered to belong to the same grid row. Such selection involves a deformation with respect to the classic rectangular grid. We are taking advantage of function U invariance:

$$y = y_0 H(x) \tag{3}$$

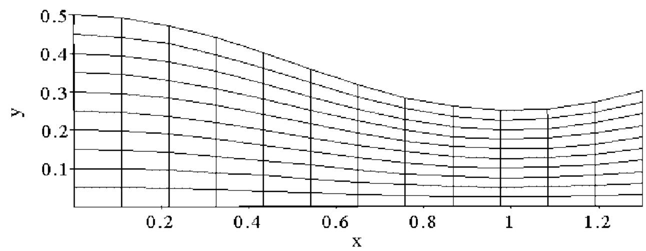

Figure 2. Control grid (13×11) for the streamfunction in the curvilinear duct.

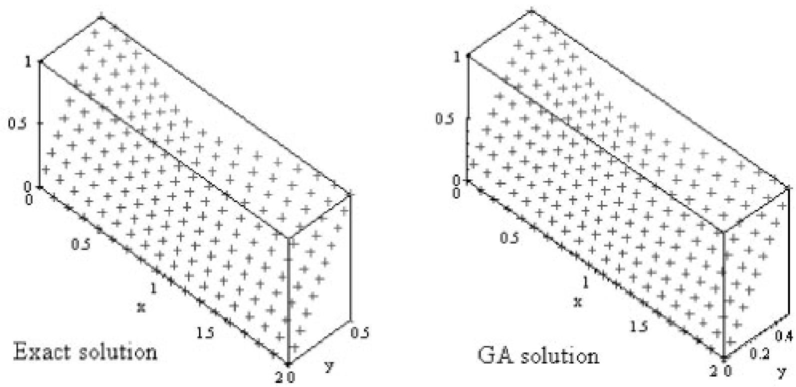

Figure 3. Comparison of the solutions for the potential flow in a duct, between the exact solution and the GA solution at 5000 generations.

The solutions of the Laplace equation have well-based properties [8]. These properties can be incorporated into the design of some characteristics of the GA. Thus, the dynamical encoding employed uses the maximum–minimum principle. A random value in the interval [0,1] is assigned to each $U(x, y)$ solution for the nodes of the grid with the only increasing monotonicity condition in y due to the symmetry problem.

Also, the mean value theorem for the harmonic functions is used in the selection stage, resulting in the numerical schema:

$$\begin{aligned} M &= \frac{1}{4}[U(x_{i+1}, y_j) + U(x_{i-1}, y_j) + U(x_i, y_{j+1}) + U(x_i, y_{j-1})] \\ f &= M - U(x, y) \\ \text{fitness function} &= \min\{|f|\} \end{aligned} \tag{4}$$

In Figure 3, the results obtained with GA in 5000 generations are compared with the analytical solution.

The results, for a fixed value of x, $x = 0.842$, and y variable, are shown in Table I and Figure 4.

Table I. Evolution of GA (13,11) Grid with $x = 0.842$ fixed.

x	y	$\boldsymbol{U(x, y)}$ **Exact**	$U(x, y)$ 3000 gen	$U(x, y)$ 5000 gen
0.842	0.000	**0.000**	0.000	0.000
0.842	0.027	**0.100**	0.097	0.098
0.842	0.053	**0.200**	0.194	0.197
0.842	0.080	**0.300**	0.292	0.296
0.842	0.106	**0.400**	0.391	0.395
0.842	0.133	**0.500**	0.490	0.494
0.842	0.159	**0.600**	0.592	0.594
0.842	0.186	**0.700**	0.693	0.696
0.842	0.212	**0.800**	0.795	0.797
0.842	0.239	**0.900**	0.897	0.898
0.842	0.265	**1.000**	1.000	1.000

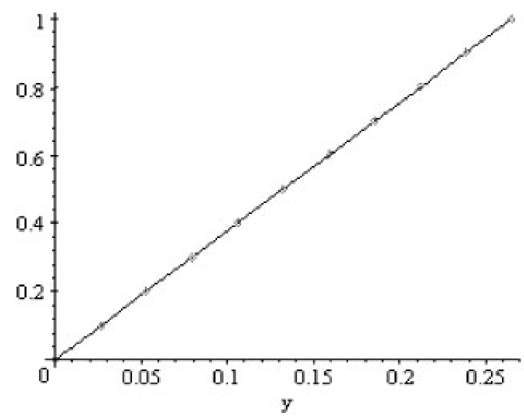

Figure 4. Comparison between the potential flow $U(x, y)$ or exact solution (solid line), and, the GA solution at 3000 generations (diamonds), $x = 0.842$.

The results of the GA at generation 3000 show acceptable values, though the results at generation 5000 seem to show an evolution towards more adapted and accurate values. Beyond 5000 generations the GA does not seem to improve the results.

3.2. Speeds for compressible flow in a nozzle

Now, the speeds for transonic flow in the compressible and isentropic flow within a nozzle, [9, 10] are studied. The difficulty of the problem is well known. Our objective is to use the evolutionary method to obtain an approximate value of the speed component in a totally developed flow. The lateral section of the nozzle for compressible flow is illustrated in Figure 5. The central line is the symmetry axis.

The resulting differential equation for the speed components is obtained from the continuity equation

$$\nabla \cdot [\rho(u, v)\mathbf{u}] = 0 \tag{5}$$

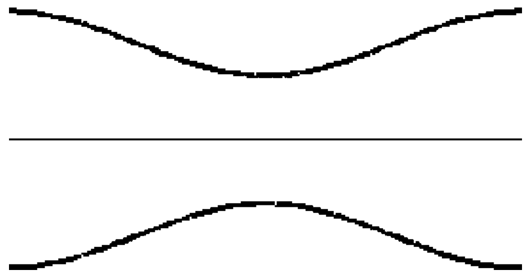

Figure 5. Lateral section of the nozzle for compressible flow. The central line is the symmetry axis.

For compressible and isentropic flow, the density depends on the speed components:

$$\rho = \rho_0 \left[1 - \frac{\gamma - 1}{2c_0^2}(u^2 + v^2) \right]^{\frac{1}{\gamma - 1}} \tag{6}$$

c_0 is the speed of sound in normal conditions. $\gamma = 1.4$ for the air.

After working with the speeds, the equation solely contains first derivatives. Making a simple change of variables, the following first-order partial differential equation results for components without dimensions $(u', v') = (u/c_0, v/c_0)$. It is a non-linear equation:

$$\left[\frac{\gamma + 1}{2} u'^2 + \frac{\gamma - 1}{2} v'^2 - 1 \right] \frac{\partial u'}{\partial x} + u'v' \frac{\partial u'}{\partial y} + u'v' \frac{\partial v'}{\partial x} + \left[\frac{\gamma + 1}{2} v'^2 + \frac{\gamma - 1}{2} u'^2 - 1 \right] \frac{\partial v'}{\partial y} = 0 \tag{7}$$

We take boundary conditions for (u', v')

$u'(0, y) = u'(2, y) = \text{Cte}$ (constant value, in the numerical application here, 0.242)

$$v'(0, y) = v'(2, y) = 0 \tag{8}$$

Free boundary conditions are introduced in the lower frontier (symmetry axis) of the nozzle, $u'(x, 0)$, $v'(x, 0)$.

The shape of the nozzle is given by $y = H(x)$, where H denotes the ratio of the half-height to the half-height at the throat, and then at the wall of the nozzle the condition that the fluid follows the wall is $\mathbf{n} \cdot (u'(x, H(x)), v'(x, H(x))) = 0$, with $\mathbf{n}$ the unitary surface vector of the wall. Then $v'(x, H(x))/u'(x, H(x)) = \mathrm{d}H(x)/\mathrm{d}x$.

Also, in this problem, $\mathbf{curl}(u', v') = 0$. We included this information in the evolutionary process through the condition

$$\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = 0 \tag{9}$$

With boundary conditions (8) and the frontier conditions, the solution has axial symmetry and thus the area from the symmetry axis till the upper wall of the nozzle is sufficient to be considered as a domain of the problem.

3.3. Grid evolution

In this two-dimensional domain a grid is generated, with variable steps h_x and h_y. A numerical scheme that proved effective in the interior of the domain for getting an acceptable evolution of the grid points was the centred scheme:

$$\frac{\partial u_{i,j}}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h_x} \tag{10}$$

Analogous for the y derivative. Other tested schemata, progressive as well as regressive, produce the convergence of grid values toward a constant value that depends on the scheme.

For avoiding all the grid points tending toward a constant, schemata of two steps were used at the points of the lower frontier with free boundary conditions and progressive schemata. Thus, the derivative for v in the lower frontier is

$$\frac{\partial v_{i,1}}{\partial y} = \frac{v_{i+1,1} - v_{i,1}}{h_y} \tag{11}$$

A value for u as well as for v in a chromosome of two criteria, $(u'(x,y), v'(x,y))$, is associated to each grid point (x,y). 10 bits are used for each criterion. Thus, the chromosome length is 20 bits.

The initial candidate solutions for $(u'(x,y), v'(x,y))$ to start the evolution process should contain the qualitative or general characteristics of the exact solution to get an appropriate convergence towards the real solution. Thus, we consider the solution in the inferior frontier, $y=0$, to be growing until $x=1$ from the frontier condition at $x=0$, then we take it to decrease until the exit frontier, $x=2$, and here it takes the constant value as at $x=0$, and to each x fixed, a random value is assigned to the points that have the same value as x, in an interval close to the value given to the node $(x,0)$. All initial solutions of the grid are stored in memory. The following sequence is repeated until the stop criterion:

- 10 points of the grid are chosen at random.
- A chromosome is chosen as a trial solution for each point, according to the interval associated with the grid point in the current generation (dynamic interval).
- 10 chromosomes are obtained from the population.
- One point crossover between couples of chromosomes, with crossover probability $\chi = 1$.
- The offsprings are mutated, uniform mutation with high probability $\mu = 0.8$.
- Tournament selection between the offspring and the previous solution of a grid point in the sampling population. The $(u'(x,y), v'(x,y))$ selected will be the one that minimizes the numerical scheme associated with the previous differential equation. This selected individual becomes the new solution that is assigned to this grid point.

A previously sampled point cannot be taken again until the entire grid is selected.

3.4. Two-stage optimization

Due to the non-linearity of the problem, the initial approximate solution (with the complete numerical scheme) can evolve deficiently, diverging towards very distant values. To avoid this divergence the boundary conditions are used. Thus, in the first stage of the algorithm the part of the numerical scheme that only contains the function u' (ideal fluid problem with only

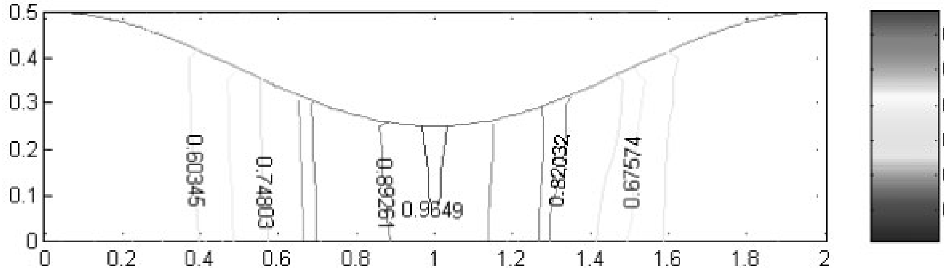

Figure 6. Mach solution for the normalized flow, after 200 000 generations.

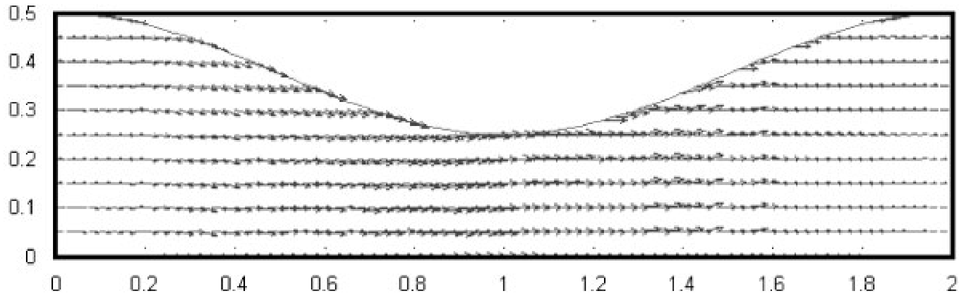

Figure 7. Velocity field, after 200 000 generations.

component in the x-axis) is chosen as objective function. The rest of the differential equation is considered as a small perturbation

$$\left(\frac{\gamma+1}{2}u'^2-1\right)\frac{\partial u'}{\partial x}=0 \tag{12}$$

After a number of generations, for example, here 2000 generations, the evolutionary system is fixed on the values of the ideal approximation. Now, the second stage starts, where the full numerical scheme of the differential equation is used. Empirical checking confirms that this strategy of two stages obtains more optimum outcomes than the one-stage strategy. The first stage is a learning stage. The solutions obtained at this first stage are a training set for starting the next stage, which is now in 2D context.

In Figures 6 and 7 are illustrated the solution obtained with this evolutionary method after 200 000 generations and a total number of 1339 nodal points are illustrated. They show an acceptable approximation to the qualitative shape of the real solution for this problem. The values of the solution shown here are averages over ten executions of the algorithm.

The computed results are in acceptable agreement with the numerical results obtained from the linearization algorithm of Gelder [6, 11] differences are around 5%.

4. CONCLUSIONS

The most important contribution in this paper is the demonstration of the capability and applicability of the Evolutionary Algorithms such as Genetic Algorithms for solving linear and non-linear boundary problems of stationary potential flow. Pioneering results were presented in References [4, 7], and here the work has been reoriented to fluid dynamics problems and extended to non-linear potential flow problems inside a nozzle.

We have proposed a genetic algorithm that through a partial sampling technique and a dynamic interval reduction permits great advantages with respect to other evolutionary algorithms like avoiding a rigid connectivity to discretize the domain and thus it is a meshless method. The amount of the computer storage is low and convergence behaviour is good because of taking into account in the algorithm the qualitative characteristics of the solution. Also, the method is easy to implement in parallel environments. However, more analysis must be accomplished to improve accuracy of the numerical results. Many directions, like use of higher-order approximation schemes into the objective function, are open. The proposed algorithm does not increase much the total computational cost and neither memory requirements.

REFERENCES

1. Bäck T, Fogel DB, Michalewicz Z. *Handbook of Evolutionary Computation*. New YorK: Oxford IOP. 1997.
2. Holland JH. Adaptation in natural and artificial systems. In *Bradford Books* (Reprint edn). MIT Press: Cambridge, MA, 1992.
3. Vose MD. The simple genetic algorithm. Foundations and theory. In *Bradford Book*. MIT Press: Cambridge, MA, 1999.
4. Jiménez JA, Abderramán JC, Cuesta PD, Winter G. Solving Elliptical PDE in a NATURAL way. *Proceedings of the Genetic and Evolutionary Computation Conference*, *GECCO-2001*. Morgan Kaufmann: San Mateo, CA, 2001; 1307–1313.
5. Caruana RA, Schaffer J. Representation and hidden bias: Gray vs binary coding for genetic algorithms. *Proceeding of* 5° *I.C. Machine Learning*. Morgan Kaufmann: San Mateo, CA, 1988.
6. Gelder D. Solution of the compressible flow equations. *International Journal for Numerical Methods in Engineering* 1987; (3):35–43.
7. Winter G, Jiménez JA, Abderramán JC, Cuesta PD. *Solving Laplace PDE by Genetic Algorithms*. III Ingenet Workshop. Las Palmas: Spain, 2000.
8. Weinberger HF. *A first course in Partial Differential Equations*. Dover: New York, 1995.
9. Shapiro AH. *The Dynamics and Thermodynamics of Compressible Fluid Flow*, vol. 2. Wiley: New York, 1953.
10. Landau LD, Lifshitz EM. *Fluid Mechanics* (2nd edn). Butterworth-Heinemann: Oxford, 1995.
11. Pironneau O. *Methods des Élements Finis pour les fluides*. Masson: Paris, 1988.

